

LOW GRAVITY LIQUID SLOSHING IN AN ARBITRARY AXISYMMETRIC TANK PERFORMING TRANSLATIONAL OSCILLATIONS

by

Wen-Hwa Chu

Technical Report No. 4

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George C. Marshall Space Flight Center
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SOUTHWEST RESEARCH INSTITUTE
SAN ANTONIO HOUSTON

SOUTHWEST RESEARCH INSTITUTE
8500 Culebra Road, San Antonio, Texas 78206

Department of Mechanical Sciences

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H. N. Abramson

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ABSTRACT

The title problem is solved by using characteristic functions. The force and moment due to lateral translation of a rigid axisymmetric tank are obtained. A mechanical model is also proposed. However, no numerical example is given because of the considerable programming effort required.

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NOTATIONS

a	radius of the contact circle of the liquid with the tank
C	contact curve of F and W
f	equilibrium (mean) interface elevation
F	equilibrium (mean) interface or f/a .
F_e	(exact) interface
$F_{av.}$	average interface height from origin (Fig. 1)
g	effective gravitational acceleration
$G(z, \theta)$	wetted wall, $r/a - G = 0$ on W
h	interface perturbation
h_0	a reference length, say l_I (Fig. 1) or l_{II}
$l_{av.}$	average liquid height from the center of the bottom
M_F	liquid mass, $\rho V a^3$
n	outer normal
N	an interger
N_B	Bond number, $\rho g a^2 / \sigma$
p	pressure
p_I	equilibrium liquid pressure at origin (just below interface), a constant
p_u	ullage pressure
p_{uI}	equilibrium ullage pressure at origin (just about interface), a constant
P_k	$p_{uI} / \rho w_k^2 a$

NOTATIONS (Cont'd)

q	magnitude of velocity
\vec{q}	velocity vector
r, θ, z	cylindrical coordinates
S	surface area nondimensionalized by a^2
t	time
T	contact curve of F_e and W_e
\vec{T}	surface tension force
u, v, w	velocity components along x, y, z
V	volume of the liquid divided by a^3
W	wetted wall bounded by $F, r/a - W(R, \theta) = 0$ on W
W_e	exact wetted wall bounded by F_e
x_0, y_0, z_0	amplitude of tank displacement along x, y, z
X, Z	$x/a, z/a$, respectively
Γ	hysteresis coefficient, Equation (28)
δ_{ij}	Kronecker delta
$\Delta\rho$	density difference between lower and upper fluid, $\rho - \rho_u$
ζ	interface elevation, $f + h$
θ_F	angle between normal to F and z -axis
κ	the mean curvature, Equation (6)
κ_1, κ_2	principal curvatures
ρ	lower fluid density
ρ_u	density of ullage fluid (vapor or gas)

NOTATIONS (Cont'd)

σ	surface tension
ω	frequency of oscillation
ω_k	k^{th} natural frequency
Ω^2	$\rho a^3 \omega^2 / \sigma$, product of Bond number and frequency parameter

SUPERSCRIPTS AND SUBSCRIPTS

$(\quad)_I$	(\quad) at the vertex of the equilibrium interface (origin)
$(\quad)_{II}$	(\quad) at the equilibrium contact point
$(\quad)^{\sim}$	the amplitude of (\quad)
$(\quad)^{\rightarrow}$	(\quad) is a vector
$(\quad)^{\hat{}}$	(\quad) is a unit vector
$(\quad)_C$	(\quad) on C
$(\quad)_{c.g.}$	(\quad) related to center of gravity
$(\quad)_F$	(\quad) on F
$(\quad)_m$	(\quad) associated with $\cos(m\theta)$ circumferential mode
$(\quad)_T$	(\quad) on T
$(\quad)_W$	(\quad) on W
$(\quad)_{W_e}$	(\quad) on W_e

INTRODUCTION

The behavior and consequences of fuel sloshing in rockets under a high effective gravity are recognized problems which have been quite well understood (Refs. 1, 2, 3). The problem of low gravity fuel sloshing, characterized by the significant role of interfacial tension, is now a subject of importance for application to coasting rockets or orbital stations.

The equilibrium behavior of fluids at zero and/or low gravity has been studied in References 4 through 7. The theoretical determination of an equilibrium interface shape is nonlinear and requires a trial and error procedure (Refs. 5, 6). Satterlee and Reynolds (Ref. 8) have successfully solved the free sloshing problem in cylindrical containers under low gravity and formulated a variational principle for this purpose. More recently, Dodge and Garza (Ref. 9) performed experiments under simulated low gravity conditions and predicted the sloshing forces for a circular cylindrical tank under lateral excitation. An equivalent mechanical model was also given.

For an "arbitrary" tank under normal gravity, the fuel sloshing problem has been investigated in References 10 through 13. The object of the present report is to predict forces exerted on an arbitrary axial symmetric tank (rigid) subject to lateral excitation in a low gravity condition.

FORMULATION OF PROBLEM

Governing Equations

Assume the tank is undergoing translational oscillations. The momentum equations in a tank fixed coordinate system are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \hat{x}_o \omega^2 \cos(N\omega t) \quad (1a)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \hat{y}_o \omega^2 \cos(N\omega t) \quad (1b)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \hat{z}_o \omega^2 \cos(N\omega t) - g \quad (1c)$$

Assume irrotational flow in the moving coordinates. There exists a velocity potential such that

$$\nabla \phi = \vec{q} \quad (2)$$

The continuity and irrotationality equation yields

$$\nabla^2 \phi = 0 \quad (3)$$

inside the liquid.

Integration of the momentum equation yields the unsteady Bernoulli equation:

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \frac{p - p_I}{\rho} - \hat{x}_O \omega^2 x \cos(N\omega t) - \hat{y}_O \omega^2 y \cos(N\omega t) \\ - \hat{z}_O \omega^2 z \cos(N\omega t) + gz = 0 \end{aligned} \quad (4)$$

At the interface, the pressure-difference curvature relation is (Lamb, pp. 456 and 265)

$$p - p_u = \sigma(\kappa_1 + \kappa_2) \quad \text{on } F_e \quad (5)$$

where the curvature is negative if the center of the curvature is at the upper side. In cylindrical coordinates (Ref. 8, p. 25),

$$\begin{aligned} \kappa = \kappa_1 + \kappa_2 = - \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial \zeta}{\partial r}}{\left[1 + \left(\frac{\partial \zeta}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \zeta}{\partial \theta} \right)^2 \right]^{1/2}} \right. \\ \left. + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\frac{\partial \zeta}{\partial \theta}}{\left[1 + \left(\frac{\partial \zeta}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \zeta}{\partial \theta} \right)^2 \right]^{1/2}} \right\} \end{aligned} \quad (6)$$

The resultant interface dynamic condition from Equations (4), (5), and (6) is then

$$\begin{aligned} - \rho \frac{\partial \phi}{\partial t} - \frac{1}{2} q^2 - (p_{u_I} - p_I) - (\Delta \rho) g \zeta + \sigma \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial \zeta}{\partial r}}{\left[1 + \left(\frac{\partial \zeta}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \zeta}{\partial \theta} \right)^2 \right]^{1/2}} \right. \\ \left. + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\frac{\partial \zeta}{\partial \theta}}{\left[1 + \left(\frac{\partial \zeta}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \zeta}{\partial \theta} \right)^2 \right]^{1/2}} \right\} + \rho \hat{x}_O \omega^2 x \cos(N\omega t) + \\ + \rho \hat{y}_O \omega^2 y \cos(N\omega t) + \rho \hat{z}_O \omega^2 z \cos(N\omega t) = 0 \end{aligned} \quad (7)$$

Neglect flow in the upper fluid (vapor or gas)

$$p_u = p_{uI} - \rho_u g z \quad (8)$$

Under equilibrium conditions, assuming the interface is axisymmetric,

$$-(p_{uI} - p_I) - (\Delta\rho)gf + \sigma \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial f}{\partial r}}{\left[1 + \left(\frac{\partial f}{\partial r} \right)^2 \right]^{1/2}} \right\} = 0 \quad (9a)$$

Since the origin is at point I (Fig. 1) where $f = 0$, then

$$-(p_{uI} - p_I) = -\sigma \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial f}{\partial r}}{\left[1 + \left(\frac{\partial f}{\partial r} \right)^2 \right]^{1/2}} \right\} = \sigma(\kappa_1 + \kappa_2)_I = \sigma\kappa_I \quad (9b)$$

The interface kinematic condition (Ref. 12, pp. 7 and 9) is

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial n} \left[1 + \left(\frac{\partial \zeta}{\partial r} \right)^2 + \left(\frac{\partial \zeta}{r \partial \theta} \right)^2 \right]^{1/2} \quad \text{on } F_e \quad (10)$$

On the wetted wall,

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } W \quad (11)$$

The solution is governed by Equation (3) subject to boundary conditions, Equations (7), (10), (11). In addition, there is a contact angle condition, the linearized form of which will be given later [Eq. (24)].

Linearization

In the formulation of problems, only the free surface conditions are nonlinear. For small oscillations, the problem can be linearized. Subtracting Equation (8) from Equation (7), the linearized interfacial dynamic equation can be easily shown to be

$$\begin{aligned}
 & -\rho \frac{\partial \phi}{\partial t} - (\Delta \rho)gh + \sigma \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial h}{\partial r}}{\left[1 + \left(\frac{\partial f}{\partial r}\right)^2\right]^{3/2}} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\frac{\partial h}{\partial \theta}}{\left[1 + \left(\frac{\partial f}{\partial r}\right)^2\right]^{1/2}} \right\} + \\
 & + \rho \hat{x}_0 \omega^2 x \cos(N\omega t) + \rho \hat{y}_0 \omega^2 y \cos(N\omega t) + \rho \hat{z}_0 \omega^2 z \cos(N\omega t) = 0 \quad (12)
 \end{aligned}$$

where the definitions of h , f , and other quantities are shown in Figure 1.

In this report, the investigation will be limited to lateral excitations in which $\hat{y}_0 = \hat{z}_0 = 0$ and $N = 1$. Let

$$\phi = \hat{\phi} \sin(\omega t) \quad , \quad h = \hat{h} \cos(\omega t)$$

Equation (12) yields

$$\begin{aligned}
 & -\rho \omega \hat{\phi} - (\Delta \rho)g\hat{h} + \sigma \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{r \frac{\partial \hat{h}}{\partial r}}{\left[1 + \left(\frac{\partial f}{\partial r}\right)^2\right]^{3/2}} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\frac{\partial \hat{h}}{\partial \theta}}{\left[1 + \left(\frac{\partial f}{\partial r}\right)^2\right]^{1/2}} \right\} + \\
 & + \rho \hat{x}_0 \omega^2 x = 0 \quad (13)
 \end{aligned}$$

The linearized form of Equation (10) is:

$$\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial n} \left[1 + \left(\frac{\partial f}{\partial r} \right)^2 \right]^{1/2} \quad (14)$$

Let

$$\frac{\partial \phi}{\partial n} = \frac{\partial \hat{\phi}}{\partial n} \sin(\omega t) \quad (15)$$

Equations (14) and (15) yield

$$-\omega \hat{h} = \frac{\partial \hat{\phi}}{\partial n} \sqrt{1 + \left(\frac{\partial f}{\partial r} \right)^2} \quad (16)$$

The pressure under equilibrium conditions is

$$p_o = p_{uI} + \sigma(\kappa_1 + \kappa_2) - \rho_u g f \quad \text{on } F \quad (17a)$$

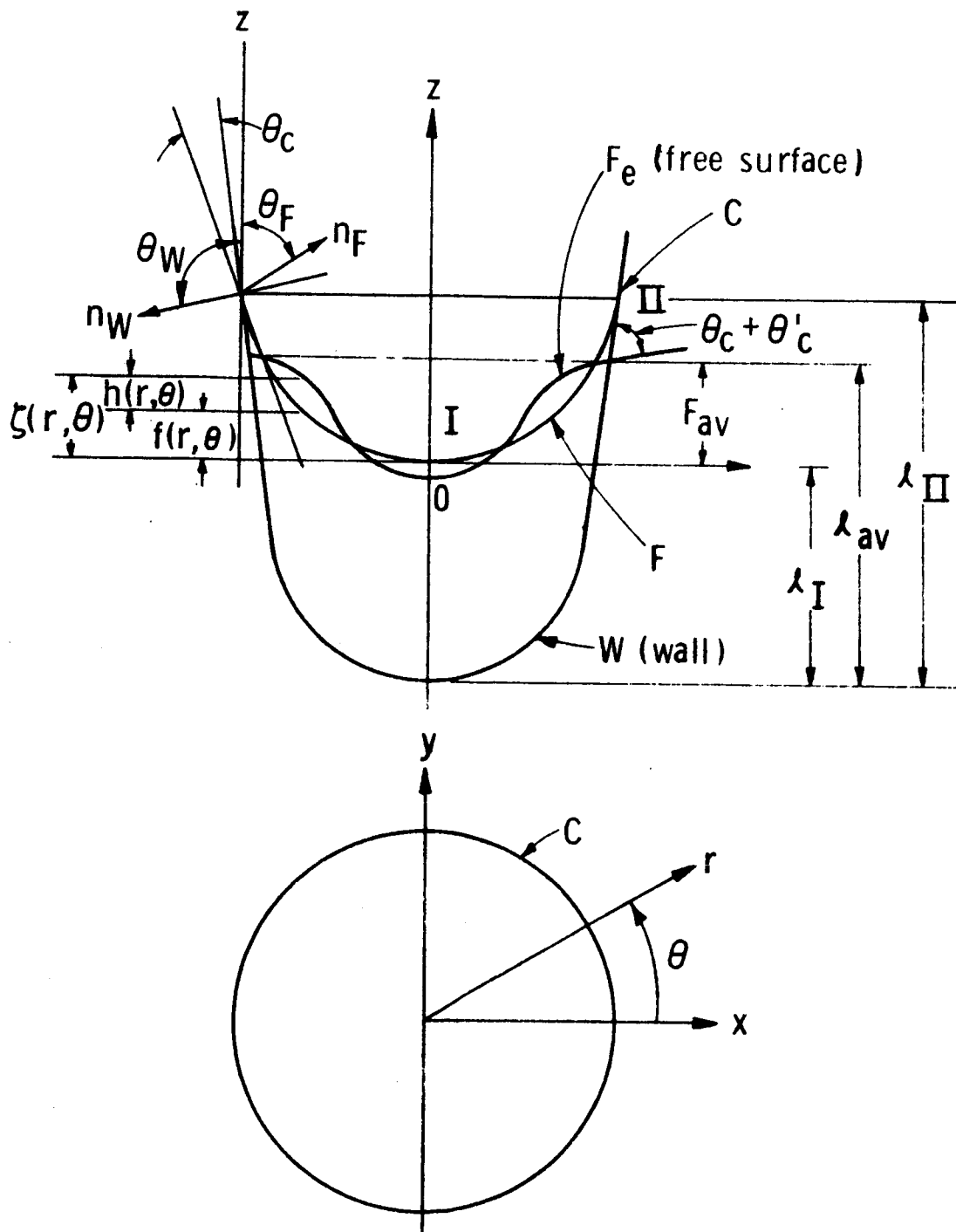
On the wall, the time independent pressure is

$$p_o = p_I - \rho g z \quad \text{on } W \text{ as well as on } W_e \quad (17b)$$

The perturbation pressure on the wall from Equation (4) after subtracting Equation (17b) is

$$p' = p - p_o \cong -\rho \frac{\partial \sigma}{\partial t} + \hat{x}_o \omega^2 x \cos(\omega t) \quad \text{on } W \quad (18)$$

Now, the contact angle correlation will be discussed. The contact angle, θ_C , is the angle between the wall and the tangent to the exact free surface, F_e , i.e. (Fig. 1):



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Figure 1. Some Nomenclature

$$\theta_C = (\pi - \theta_W) - \theta_F \quad (19)$$

The perturbation equation states

$$\theta'_C = -\theta'_F \quad (20)$$

The directional cosine of the normal to the exact interface with respect to the vertical axis is

$$\cos(\theta_F + \theta'_F) = \frac{-\frac{\partial f}{\partial r}}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} + \frac{-\frac{\partial h}{\partial r}}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} = \cos \theta_F + \theta'_C \sin \theta_F$$

Since

$$\cos \theta_F = \frac{\left(-\frac{\partial f}{\partial r}\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}}$$

then

$$\sin \theta_F = \frac{\pm 1}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} \quad (21)$$

The plus sign is taken for convex interfaces viewed from the fluid.

The perturbation contact angle is then

$$\theta'_F = \frac{1}{\sin \theta_F} \frac{\frac{\partial h}{\partial r}}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} \quad (22)$$

Presumably, the perturbation contact angle would increase with the normal velocity of the interface, and it may depend on the location of the contact point. Thus, one may assume the hysteresis condition

$$\theta'_C = \gamma_1 \frac{\partial \phi}{\partial n} - \gamma_2 h \quad (23)$$

Making use of Equation (16), Equations (22) and (23) yield

$$\frac{\partial \hat{h}}{\partial r} = \left[\gamma_1 \frac{\omega}{\sqrt{1 + \left(\frac{\partial f}{\partial r} \right)^2}} \hat{h} + \gamma_2 \hat{h} \right] \sin \theta_F \sqrt{1 + \left(\frac{\partial f}{\partial r} \right)^2} = \gamma \hat{h} \quad (24)$$

Nondimensionalization

Let

$$R = \frac{r}{a}, \quad Z = \frac{z}{a}, \quad F = \frac{f}{a}, \quad X_0 = \frac{\hat{x}_0}{a}, \quad \hat{h} = \sqrt{\frac{ag}{\omega^2}} H, \quad \hat{\phi} = \sqrt{ga^3} \Phi$$

$$U = \frac{\hat{u}}{\sqrt{ga}}, \quad \Omega^2 = \frac{\rho a^3 \omega^2}{\sigma}, \quad \Gamma = a\gamma, \quad \Gamma_1 = \gamma_1 \sqrt{ga}, \quad \Gamma_2 = \gamma a$$

and the Bond number

$$N_B = \frac{(\Delta \rho)ga^2}{\sigma} \quad (25)$$

The interfacial dynamic equation $\left[\text{Eq. (13)} \times \frac{a^2 \omega}{\sigma \sqrt{ga}} \right]$ becomes:

$$\begin{aligned}
& - \left\{ \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{R \frac{\partial H}{\partial R}}{\left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{3/2}} \right) + \frac{1}{R^2} \frac{\partial}{\partial \theta} \left(\frac{\frac{\partial H}{\partial \theta}}{\left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{1/2}} \right) \right\} + \\
& + N_B H + \Omega^2 \Phi = X_O R \cos \theta \cdot \Omega^2 \sqrt{\frac{\omega^2 a}{g}}
\end{aligned} \tag{26}$$

The interfacial kinematic condition equation states that

$$- H = \frac{\partial \Phi}{\partial n^*} \sqrt{1 + \left(\frac{\partial F}{\partial R} \right)^2} ; \quad n^* = \frac{n}{a} \tag{27}$$

The contact angle hysteresis condition reduces to

$$\frac{\partial H}{\partial R} = \Gamma H = \left[\Gamma_1 \sqrt{\frac{\omega^2 a}{g}} \sin \theta_F + \Gamma_2 \right] H \tag{28}$$

The wetted wall condition is

$$\frac{\partial \Phi}{\partial n^*} = 0 \tag{29}$$

and the governing partial differential equation is

$$\nabla^2 \Phi = 0 \tag{30}*$$

*From here on, ∇ represents gradient nondimensionalized by a .

CHARACTERISTIC FUNCTION METHOD

Properties of Characteristic Functions

The program for an arbitrary axisymmetric tank will be solved by the characteristic function method. The characteristic function satisfies the Laplace equation

$$\nabla^2 \Psi = 0 \quad (31)$$

subject to the condition that

$$\frac{\partial \Psi}{\partial n^*} = 0 \quad \text{on } W \quad (32)$$

and

$$\frac{\partial \Psi}{\partial n^*} = \lambda \Psi \quad \text{on } F \quad (33)$$

For a general tank, the characteristic functions may not and need not be the normal modes. From the divergence theorem and Equations (31), (32) and (33), it can be easily shown that the following orthogonal condition holds:

$$\begin{aligned} \int_F \Psi_i \Psi_j dS &= \frac{1}{\lambda_i - \lambda_j} \int_F [\lambda_i \Psi_i \Psi_j - \lambda_j \Psi_j \Psi_i] dS \\ &= \frac{1}{\lambda_i - \lambda_j} \int_{F+W} \left[\Psi_j \frac{\partial \Psi_i}{\partial n^*} - \Psi_i \frac{\partial \Psi_j}{\partial n^*} \right] dS \\ &= \frac{1}{\lambda_i - \lambda_j} \int_V [\nabla \cdot (\Psi_j \nabla \Psi_i) - \nabla \cdot (\Psi_i \nabla \Psi_j)] = 0, \text{ if } i \neq j. \end{aligned} \quad (34)$$

Let

$$K_j = - \frac{\partial \Psi_j}{\partial n} \quad \text{on } F \quad (35)$$

One has an alternative form

$$\int_F \Psi_i K_j \, dS = \delta_{ij} \int_F \Psi_j K_j \, dS \quad (36)$$

which was given in Reference 8.

If the velocity potential is expanded in terms of characteristic functions, one has

$$\Phi = \sum c_j \Psi_j \quad (37)$$

From Equations (35) and (27),

$$H = \sum C_j \left[-\lambda_j \Psi_j \sqrt{1 + \left(\frac{\partial \Phi}{\partial R} \right)^2} \right] = \sum C_j K_j \quad (38)$$

which is again a familiar equation given in Reference 8.

It is important to note that the characteristic functions and their derivatives can be calculated by finite difference method (Method II, Ref. 15) to evaluate integrals and constants entering the force and moment formulas given later. *

*Upon completion of the present report, the following paper by Gordon C. K. Yeh was published in which the Ritz Method was proposed to calculate the characteristic functions: "Free and Forced Oscillations of a liquid in an axisymmetric tank at low gravity environments, " J. Appl. Mech., Mar. 1967. This paper does not give an expression for the force response nor any mechanical model.

Free Oscillations of an Arbitrary Axisymmetric Tank

To include the breathing vibration in an elastic tank at the same time, it is assumed that

$$\Phi \propto \cos (m\theta) \quad , \quad H \propto \cos (m\theta) \quad (39)$$

The interface dynamic condition, Equation (26), for $X_0 = 0$ reduces to

$$-\left\{ \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{R \frac{\partial H}{\partial R}}{\left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{3/2}} \right) - \frac{m^2 H}{R^2 \left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{1/2}} \right\} + N_B H + \Omega^2 \Phi = 0 \quad (40)$$

Application of the series expansions given by Equations (37) and (38) and the Galerkin method yield the following matrix equation:

$$\left\{ \left(-\Gamma[\nu_{m_{ij}}] - [\gamma_{m_{ij}}] + \frac{m^2}{R^2} [\epsilon_{m_{ij}}] + N_B [\beta_{m_{ij}}] \right) - \Omega^2 [\delta_{ij}] \right\} [c_j] = 0 \quad (41)*$$

where the matrix elements are given in Tables 1 and 2.

Equation (41) takes the form

$$\{[A] - \Omega^2[I]\}[C] = 0 \quad (42)$$

The eigenvalue problem of Equation (42) can be solved on a high speed computer. The lowest few eigenvalues yield the lowest few natural

It is assumed that $\frac{\partial K_{m_j}}{\partial n^} = \Gamma K_{m_j}$ corresponding to the constant angle condition, Equation (28).

TABLE 1. TABLE OF CONSTANTS

$$b_k = -\pi e_k^* \sum_j c_{kj} (K_{m_j})_{II}$$

$$d_k: \text{ see Equation (56)}$$

$$e_k^*: \text{ see Equation (58)}$$

$$\ell_k = e_k^* \sum_j c_{kj} \mu_j \frac{1}{2} a_{m_j}^2$$

$$t_k = e_k^* \sum_j c_{kj} (\tilde{k}_j + \lambda_j g_j) \frac{1}{2} a_{m_j}^2$$

$$\nu_1 = F_{II} - \int_0^1 2FR \, dR$$

$$\nu_2 = \frac{\pi}{2} F_{II}^2 - \frac{1}{2} \int_F -F^2 \frac{dS}{\sqrt{1 + \left(\frac{\partial}{\partial r}\right)_{II}^2}} + \frac{1}{3} \pi$$

$$\nu_3 = F_{II} \nu_6 - \left\{ \frac{1 + \Gamma}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2}} - \frac{\Gamma \left(\frac{\partial F}{\partial R}\right)_{II}^2}{\left[1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2\right]^{3/2}} \right\}$$

$$\nu_4 = -\nu_6 + a\kappa_I$$

$$\nu_5 = \pi \nu_3 + a\kappa_I \left(F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right)$$

$$\nu_6 = \frac{\Gamma \left(\frac{\partial F}{\partial R}\right)_{II}}{\left[1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2\right]^{3/2}} + \frac{\left(\frac{\partial F}{\partial R}\right)_{II} + \left(\frac{\partial G}{\partial Z}\right)_{II} \left[1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2\right]}{\left[1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2\right]^{1/2}}$$

$$\nu_{m_{ij}} = \frac{2\pi}{a_{m_j}^2} \left[\frac{\Psi_{m_i} \Psi_{m_j}}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)_{II}^2}} \right]$$

TABLE 2. TABLE OF INTEGRALS

$$g_j = \frac{1}{a_{m_j}^2} \int_F R \Psi_{m_j} dS$$

$$\tilde{k}_j = \frac{1}{a_{m_j}^2} \int_F \Psi_{m_j} \frac{\frac{\partial F}{\partial R}}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2}} dS$$

$$a_{m_j}^2 = \int_F \Psi_{m_j}^2 dS$$

$$\beta_{m_{ij}} = \frac{1}{a_{m_j}^2} \int_F \sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2} \Psi_{m_i} \Psi_{m_j} dS$$

$$\begin{aligned} \gamma_{m_{ij}} = \frac{1}{a_{m_j}^2} & \left\{ \int_F \frac{1}{1 + \left(\frac{\partial F}{\partial R}\right)^2} \frac{\partial \Psi_{m_i}}{\partial R} \frac{\partial \Psi_{m_j}}{\partial R} dS \right. \\ & + 2 \int_F \frac{\frac{\partial F}{\partial R} \frac{\partial^2 F}{\partial R^2}}{\left[1 + \left(\frac{\partial F}{\partial R}\right)^2\right]^2} \left[\Psi_{m_j} \frac{\partial \Psi_{m_i}}{\partial R} + \Psi_{m_i} \frac{\partial \Psi_{m_j}}{\partial R} \right] dS \\ & \left. + \frac{\left(\frac{\partial F}{\partial R} \frac{\partial^2 F}{\partial R^2}\right)^2}{\left[1 + \left(\frac{\partial F}{\partial R}\right)^2\right]^2} \Psi_{m_i} \Psi_{m_j} dS \right\} \end{aligned}$$

$$\epsilon_{m_{ij}} = \frac{1}{a_{m_j}^2} \int_F \frac{1}{R^2} \Psi_{m_i} \Psi_{m_j} dS$$

$$\mu_j = \frac{1}{a_{m_j}^2} \int_W \Psi_{m_j} \left[R + W \frac{\partial W}{\partial R} \right] R dR d\theta \quad \text{if } \frac{Z}{a} - W(R) = 0 \text{ or } W$$

$$\mu_j = \frac{1}{a_{m_j}^2} \int_W \Psi_{m_j} \left[Z + G \frac{\partial G}{\partial Z} \right] G dZ d\theta \quad \text{if } \frac{r}{a} - G(Z) = 0 \text{ or } W$$

frequencies, and the corresponding eigenvectors yield the corresponding normal modes.

Forced Lateral Oscillations (Rigid Tank)

First, solutions will be obtained by direct expansions in characteristic functions. Let

$$\Phi = \sum_j \tilde{d}_j \Psi_{m_j} \cos(m\theta) \quad (43)$$

$$H = \sum_j \tilde{d}_j K_{m_j} \cos(m\theta) \quad (44)$$

For lateral oscillations of a rigid axisymmetric tank, $m = 1$. Analogous to Equation (41), Equation (26) and Galerkin's method yield

$$\left\{ -[A] + \Omega^2 [I] \right\} [\tilde{d}_j] = X_o \sqrt{\frac{\omega^2 a}{g}} \Omega^2 \left[\frac{1}{2} g_j \right] \quad (45)$$

where

$$A_{ij} = -\Gamma \nu_{m_{ij}} + \gamma_{m_{ij}} + m^2 \epsilon_{m_{ij}} + N_B \beta_{m_{ij}} \quad (46)$$

The coefficients \tilde{d}_j 's can be obtained by matrix inversion; however, to avoid a limiting process at zero gravity, the following nondimensional velocity potential and surface displacement are introduced:

$$\phi^* = \frac{\hat{\Phi}}{\omega a^2} = \sqrt{\frac{g}{\omega^2 a}} \Phi = \sum_j \tilde{e}_j \Psi_{m_j} \cos(m\theta) \quad (47)$$

$$h^* = \frac{\hat{h}}{a} = \sqrt{\frac{g}{\omega^2 a}} H = \sum_j \tilde{e}_j K_{m_j} \cos(m\theta) \quad (48)$$

where, from Equation (45),

$$[\tilde{e}_j] = \sqrt{\frac{g}{\omega^2 a}} [\tilde{d}_j] = \frac{X_o \Omega^2}{2} \left\{ -[A] + \Omega^2 [I] \right\}^{-1} [g_j] \quad (49)$$

Since in the neighborhood of a natural frequency the determinant $\left| [A] - \Omega^2 [I] \right|$ approaches zero, the corresponding matrix in Equation (49) may be ill-conditioned. Therefore, as an alternate, the solution will be obtained by expansion into normal modes. In the neighborhood of first natural frequency, one sloshing normal mode may be sufficient. Near higher modes, one resonant mode plus the fundamental mode may be sufficient as is the case for sloshing under normal gravity.

Let the k^{th} normal mode be

$$\Phi = \sum_j c_{k_j} \Psi_j = \sum_j c_{k_j} \Psi_{m_j} \cos(m\theta) \quad (50)$$

$$H = \sum_j c_{k_j} K_j = \sum_j c_{k_j} K_{m_j} \cos(m\theta) \quad (51)$$

where c_{k_j} are determined in free oscillations. Then, expansion in normal modes yields

$$\Phi = \sum_k d_k \Phi_k = \sum_k d_k \sum_j c_{k_j} \Psi_j = \sum_j \tilde{d}_j \Psi_j \quad (52)$$

$$H = \sum_k d_k H_k = \sum_k d_k \sum_j c_{kj} K_j = \sum_j \tilde{d}_j K_j \quad (53)$$

where

$$\tilde{d}_j = \sum_k c_{kj} d_k \quad (54)$$

By definition of normal modes, Equation (26) yields

$$\sum_k d_k [-\Omega_k^2 \phi_k + \Omega^2 \Phi_k] = X_o \Omega^2 \sqrt{\frac{\omega_a^2}{g}} R \cos \theta \quad (55)$$

By Galerkin's method,

$$d_k = \frac{-X_o \Omega^2 \sqrt{\frac{\omega_a^2}{g}}}{\Omega_k^2 - \Omega^2} \sum_j \frac{1}{2} g_j c_{kj} \quad (56)$$

The corresponding velocity potential is

$$\phi = -\omega a^2 X_o \sum_k \frac{e_k^* \Omega^2}{\Omega_k^2 - \Omega^2} \Phi_k \sin(\omega t) \quad (57)$$

where

$$e_k^* = \sum_j \frac{1}{2} c_{kj} g_j \quad (58)$$

The corresponding surface elevation is

$$h = -aX_o \sum_k \frac{e_k^* \Omega^2}{\Omega_k^2 - \Omega^2} H_k \cos(\omega t) \quad (59)$$

while

$$\tilde{e}_j = -X_o \Omega^2 \sum_k \frac{e_k^* c_{kj}}{\Omega_k^2 - \Omega^2} \quad (59a)$$

Force on the Tank

The divergence theorem will be used repeatedly to convert the surface integral on the (mean) wetted surface, W , to a volume integral minus the surface integral on the (mean) free surface, F . The force due to pressure on $W_e - W$ will be evaluated last.

The force due to hydrostatic pressure is

$$\begin{aligned} F_{x_1} &= - \int_W \rho g z \cos(n, x) a^2 dS \quad ; \quad \cos(n, x) = \frac{\partial x}{\partial n} \\ &= - \left\{ \int_V \rho g \nabla \cdot (Z \nabla X) a^3 dV - \int_F \rho g z \cos(n, x) a^2 dS \right\} \\ &= - \int_F \rho g f(R) \frac{\frac{\partial F}{\partial R} \cos \theta}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2}} \sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2} a^2 R dR d\theta = 0 \end{aligned}$$

The force due to inertia is:

$$\begin{aligned}
\hat{F}_{x_2} &= \int_W \rho \hat{x}_o \omega^2 x \cos(n, x) a^2 dS \\
&= \rho \hat{x}_o \omega^2 a^3 \int_V \nabla \cdot (X \nabla X) dV - \int_F X \cdot \frac{-\frac{\partial F}{\partial R} \cos \theta}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2}} a^3 dS \\
&= \rho X_o a^4 \omega^2 V + \rho X_o \omega^2 a^4 \pi \nu_1
\end{aligned} \tag{60}$$

Due to impulsive pressure, similarly,

$$\begin{aligned}
\hat{F}_{x_3} &= \int_W -\rho \omega \hat{\phi} \cos(n, x) a^2 dS \\
&= -\rho \omega^2 a^2 \sum_k \frac{e_k^*}{\Omega_k^2 - \Omega^2} a^2 \int_W \Phi_k \cos(n, x) dS \cdot (-\Omega^2 X_o) \\
&= \Omega^2 X_o \rho \omega^2 a^2 \sum_k \frac{t_k}{\Omega_k^2 - \Omega^2}
\end{aligned} \tag{61}$$

where

$$t_k = e_k^* \sum_j c_{kj} (\tilde{k}_j + \lambda_j g_j) \frac{1}{2} a_{mj}^2 \tag{61a}$$

In deriving Equation (61), use is made of

$$\int_V \frac{\partial \Psi_j}{\partial X} dV = \int_V \nabla \cdot (X \nabla \Psi_j) dV = \int_F X \frac{\partial \Psi_j}{\partial n} dS = \frac{1}{2} \lambda_j g_j a_{mj}^2 \tag{62}$$

Using Equations (4) and (9b), the linearized force on the wetted surface

$W_e - W$ is:

$$\begin{aligned}
F_{x_a} &= \int_{W_e - W} p \cos(n, x) a^2 dS \cong \cos(\omega t) \pi [p_{u_I} + \sigma \kappa_I \\
&\quad - \rho g F_{II} a] a^2 \sum_j \tilde{e}_j K_{mjII} \\
&= F_{x_5} + F_{x_6} + F_{x_7}
\end{aligned} \tag{63}$$

$$\hat{F}_{x_5} = (a \kappa_I) \rho \omega^2 a^4 X_O \left[\sum_k \frac{b_k}{\Omega_k^2} + \sum_k \frac{\frac{b_k}{\Omega_k^2}}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right] \tag{64}$$

$$\hat{F}_{x_6} = - \rho \omega^2 a^4 X_O F_{II} \sum_k \frac{b_k g}{\omega_k^2 a} - \rho \omega^2 a^4 X_O F_{II} \frac{\left(\frac{g}{\omega_k^2 a}\right) b_k}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \tag{65}$$

$$F_{x_7} = \rho \omega^2 a^4 X_O \left[\sum_k b_k P_k + \sum_k \frac{b_k P_k}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right] \tag{66}$$

Next, the direct force due to surface tension will be evaluated.

Let the surface tension be pulling downward. The directional cosines of the surface tension at the wall with respect to cylindrical coordinates are (Appendix A)

$$\ell \cong \left[\frac{-1}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2 + 2 \frac{\partial F}{\partial R} \frac{\partial h^*}{\partial R}}} \right]_C \tag{67a}$$

$$m \cong \left[\frac{\frac{\partial h^*}{R \partial \theta} \left(\frac{\partial F}{\partial R} + \frac{\partial G}{\partial Z} \left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right] \right)}{\sqrt{1 + \left(\frac{\partial F}{\partial R} \right)^2 + 2 \frac{\partial F}{\partial R} \frac{\partial h^*}{\partial R}}} \right]_C \quad (67b)$$

$$n \cong \left[\frac{\frac{\partial F}{\partial R} + \frac{\partial h^*}{\partial R}}{\sqrt{1 + \left(\frac{\partial F}{\partial R} \right)^2 + 2 \frac{\partial F}{\partial R} \frac{\partial h^*}{\partial R}}} \right]_C \quad (67c)$$

Then, using binomial expansion and integration,

$$\begin{aligned} F_{xT} &\cong \int_T \vec{T} \cdot (\hat{r} \cos \theta - \hat{\theta} \sin \theta) \frac{ds}{r d\theta} r d\theta \\ &= - \int_0^{2\pi} a \sigma \left\{ \cos \theta \frac{1 + \left(\frac{\partial F}{\partial R} \right)^2 - \frac{\partial F}{\partial R} \frac{\partial h^*}{\partial R}}{\left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{3/2}} + \right. \\ &\quad \left. + \sin \theta \frac{\left\{ \frac{\partial F}{\partial R} + \frac{\partial G}{\partial Z} \left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right] \right\} \frac{\partial h^*}{\partial \theta}}{\left[1 + \left(\frac{\partial F}{\partial R} \right)^2 \right]^{1/2}} \right\} R d\theta \quad (68a) \end{aligned}$$

$$= - \pi a \sigma \nu_6 \sum_j e_j (K_{m_j})_{II} \quad (68b)$$

$$= - X_o \rho a^4 \omega^2 \nu_6 \left\{ \sum_k \frac{b_k}{\Omega_k^2} + \sum \frac{\frac{b_k}{\Omega_k^2}}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \right\} \quad (68c)$$

The moment about y-axis can be obtained similarly by integration

$$M_y = \int_{W_e} [pz \cos(n, x) - px \cos(n, z)] a^2 dS - \int_C \vec{T} \times \vec{r} \cdot \hat{y} ds$$

Due to limitation of space, details of the evaluation will be omitted. It is noted, however, that not all integrals can be converted to integrals on the (mean) free surface; consequently, values of the characteristic functions on the (mean) wetted wall need be calculated in the finite difference program.

To summarize, the total force and moment are

$$\begin{aligned} \hat{F}_x = & \rho X_o \omega^2 a^4 V + \pi \nu_l \rho X_o \omega^2 a^2 + \rho \omega^2 a^4 X_o \sum_k \frac{t_k \Omega^2}{\Omega_k^2 - \Omega^2} \\ & - \rho \omega^2 a^4 X_o F_{II} \sum_k \frac{b_k g}{\omega_k^2 a} - \rho \omega^2 a^4 X_o F_{II} \sum_k \frac{\left(\frac{b_k g}{\omega_k^2 a} \right)}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \\ & + X_o \rho a^4 \omega^2 \nu_4 \left[\sum_k \frac{b_k}{\Omega_k^2} + \sum_k \frac{\left(\frac{b_k}{\Omega_k^2} \right)}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \right] \\ & + X_o \rho a^4 \omega^2 \left[\sum_k b_k P_k + \sum_k \frac{b_k P_k}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \right] \end{aligned} \quad (69)$$

$$\begin{aligned}
M_y = & \rho X_o \omega^2 a^5 [V Z_{c.g.} + \nu_2] + \rho \omega^2 a^5 X_o \sum_k \frac{\ell_k}{\left(\frac{\omega_k^2}{\omega^2}\right) - 1} \\
& + \rho X_o \omega^2 a^5 \left\{ F_{II} \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right] \sum_k \frac{g b_k}{\omega_k^2 a} + \right. \\
& \left. + F_{II} \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right] \sum_k \frac{\left(\frac{b_k g}{\omega_k^2 a}\right)}{\left(\frac{\omega_k^2}{\omega^2}\right) - 1} \right\} \\
& - \rho X_o \omega^2 a^5 \nu_5 \left\{ \sum_k \frac{b_k}{\Omega_k^2} + \sum_k \frac{\frac{b_k}{\Omega_k^2}}{\left(\frac{\omega_k^2}{\omega^2}\right) - 1} \right\} \\
& - \rho X_o \omega^2 a^5 \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right] \left\{ \sum_k b_k P_k + \sum_k \frac{b_k P_k}{\left(\frac{\omega_k^2}{\omega^2}\right) - 1} \right\} \quad (70)
\end{aligned}$$

The contained constants are given in Table 1.

MECHANICAL MODEL

The mechanical model without damping consists of a series of spring-masses located at different heights. Let the k^{th} mass be m_k , the k^{th} spring constant be $m_k \omega_k^2$, and the height of k^{th} mass-spring be z_k .

The force and moment due to the equivalent system are (Ref. 16):

$$F_x = X_o \omega^2 a M_F \left\{ \frac{m_o}{M_F} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{1}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \right\} \quad (71)$$

$$M_y = X_o \omega^2 a M_F h_o \left\{ \frac{m_o}{M_F} \frac{z_o}{h_o} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{z_k}{h_o} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{g}{\omega_k^2 h_o} + \sum_{k=1}^{\infty} \frac{\left(\frac{m_k}{M_F} \right) \left[\left(\frac{z_k}{h_o} \right) + \left(\frac{g}{\omega_k^2 a} \right) \right]}{\left(\frac{\omega_k^2}{\omega^2} \right) - 1} \right\} \quad (72)$$

For convenience, m_k can be obtained from several "masses" at different heights, i.e.,

$$m_k = \sum_n m_{k_n} \quad (73a)$$

$$m_k z_k = \sum_n m_{k_n} z_n \quad (73b)$$

It is found from comparing Equations (68) and (69) with Equations (71), (72), and (73) that

$$\begin{aligned} \frac{m_k}{M_F} = \frac{m_{k1}}{M_F} + \frac{m_{k2}}{M_F} + \frac{m_{k3}}{M_F} + \frac{m_{k4}}{M_F} = t_k - \frac{F_{II} b_k N_B}{\Omega_k^2} \\ + \nu_4 \frac{b_k}{\Omega_k^2} + b_k P_k, \quad k = 1, 2, \dots \end{aligned} \quad (74)$$

$$\begin{aligned} \frac{z_k}{h_o} = \frac{1}{\frac{m_k}{M_F}} \left\{ t_k \left[\frac{a}{h_o} \frac{l_k}{t_k} - \frac{g}{h_o \omega_k^2} \right] - \frac{F_{II} b_k N_B}{\Omega_k^2} \left[\frac{a}{h_o} \left(F_{II} \right. \right. \right. \\ \left. \left. \left. + \frac{1}{\left(\frac{\partial W}{\partial R} \right)_{II}} \right) F_{II} - \frac{g}{\omega_k^2 h_o} \right] + \nu_4 \frac{b_k}{\Omega_k^2} \left[\frac{a}{h_o} \frac{\nu_4}{\nu_5} - \frac{g}{\omega_k^2 h_o} \right] \right. \\ \left. - b_k P_k \left[\left(F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R} \right)_{II}} \right) \frac{a}{h_o} + \frac{g}{\omega_k^2 h_o} \right] \right\} \end{aligned} \quad (75)$$

where

$$\frac{g}{\omega_k^2 h_o} = \frac{a}{h_o} \frac{N_B}{\Omega_k^2} \quad (76)$$

The rigid mass m_o is given by

$$\frac{m_o}{M_F} = 1 + \pi \nu_1 - \sum_{k=1}^{\infty} \frac{m_k}{M_F} - \sum_k \frac{F_{II} b_k g}{\omega_k^2 a} + \nu_4 \sum_k \frac{b_k}{\Omega_k^2} \quad (77)$$

It is probable that

$$\pi v_1 = F_{II} \sum_k \frac{b_k g}{\omega_k^2 a} - v_4 \sum_k \frac{b_k}{\Omega_k^2} \quad (78)$$

(which cannot be proved in general) so that

$$m_o = M_F - \sum_{k=1}^{\infty} m_k \quad (79)$$

which implies that the total liquid mass equals the total mass of the model. The location of the rigid mass is

$$\begin{aligned} \frac{z_o}{h_o} = \frac{1}{\left(\frac{m_o}{M_F}\right)} & \left\{ \frac{a}{h_o} z_{c.g.} - \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{z_k}{h_o} - \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{g}{\omega_k^2 h_o} \right. \\ & + \frac{a}{h_o} \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right] F_{II} \sum_{k=1}^{\infty} \frac{F_{II} b_k N_B}{\Omega_k^2} \\ & \left. + \frac{a}{h_o} v_2 - \frac{a}{h_o} v_5 \sum_{k=1}^{\infty} \frac{b_k}{\Omega_k^2} - \frac{a}{h_o} \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R}\right)_{II}} \right] \sum_k b_k P_k \right\} \quad (80) \end{aligned}$$

It is probable but cannot be proven in general that:

$$\begin{aligned}
v_2 = \frac{h_o}{a} \left\{ - \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R} \right)_{II}} \right] F_{II} \sum_{k=1}^{\infty} \frac{F_{II} b_k N_B}{\Omega_k^2} \right. \\
+ v_5 \sum_k \frac{b_k}{\Omega_k^2} + \frac{a}{h_o} \left[F_{II} + \frac{1}{\left(\frac{\partial W}{\partial R} \right)_{II}} \right] \sum_k b_k P_k \\
\left. + \sum_k \frac{m_k}{M_F} \frac{g}{\omega_{k h_o}^2} \right\} \quad (81)
\end{aligned}$$

then

$$m_o z_o = M_F Z_{c.g.} - \sum_{k=1}^{\infty} m_k z_k \quad (82)*$$

To refer all heights to the center of the bottom, let $z_k \rightarrow z_k + \ell_{av.} - F_{av.}$

* If Equations (79) and (82) do not hold, then the mechanical model cannot duplicate the statical properties of the liquid.

CONCLUSIONS

The force and moment exerted on the tank have been obtained by using characteristic functions, and a possible mechanical model is introduced in the present report. The evaluation of eigenvalue and eigenfunctions, however, requires a finite difference computer program which has not yet been written. Such a program is needed if comparisons of the present theory are to be made with experiments.

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APPENDIX A

Directional Cosines of the Surface Tension

Let l , m , and n be the directional cosines of the surface tension at the wall (on contour T) with respect to the cylindrical coordinates r , θ , z .

Let the free surface geometric equation be

$$z - \zeta(r, \theta) = 0 \quad \text{on } F_e$$

The directional cosines of the normal to the free surface are

$$\frac{-\frac{\partial \zeta}{\partial r}}{\sqrt{1 + \left(\frac{\partial \zeta}{\partial r}\right)^2 + \left(\frac{\partial \zeta}{r \partial \theta}\right)^2}}, \quad \frac{-\frac{\partial \zeta}{r \partial \theta}}{\sqrt{1 + \left(\frac{\partial \zeta}{\partial r}\right)^2 + \left(\frac{\partial \zeta}{r \partial \theta}\right)^2}}, \quad \frac{1}{\sqrt{1 + \left(\frac{\partial \zeta}{\partial r}\right)^2 + \left(\frac{\partial \zeta}{r \partial \theta}\right)^2}}$$

First, the surface tension lies in the plane of the free surface at T ; thus,

$$\left(-\frac{\partial \zeta}{\partial r}\right)l + \left(-\frac{\partial \zeta}{r \partial \theta}\right)m + n = 0$$

Next, we shall find the tangent vector $\vec{\tau}$ of the contact curve T which is defined by

$$\mathbf{r} = \mathbf{R}_{II} + \delta(\theta), \quad \theta \equiv \theta, \quad z = \zeta(r, \theta)$$

The element of the tangent line is

$$d\mathbf{s} = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\theta}} r d\theta + \hat{\mathbf{z}} dz$$

then

$$\vec{\tau} = \frac{\frac{d\vec{s}}{rd\theta}}{\sqrt{\left(\frac{d\vec{s}}{rd\theta}\right)\left(\frac{d\vec{s}}{rd\theta}\right)}} = \frac{\hat{r} \frac{\partial \delta}{r \partial \theta} + \hat{\theta} + \left(\frac{\partial \zeta}{r \partial \theta} + \frac{\partial \zeta}{\partial r} \frac{\partial \delta}{r \partial \theta}\right) \hat{z}}{\sqrt{1 + \left(\frac{\partial \delta}{r \partial \theta}\right)^2 + \left(\frac{\partial \zeta}{r \partial \theta} + \frac{\partial \zeta}{\partial r} \frac{\partial \delta}{r \partial \theta}\right)^2}}$$

Second, the surface tension is normal to the contact curve tangent $\vec{\tau}$; thus,

$$\frac{\partial \delta}{r \partial \theta} \ell + m + \left(\frac{\partial \zeta}{r \partial \theta} + \frac{\partial \delta}{r \partial \theta} \frac{\partial \zeta}{\partial r}\right) n = 0$$

Therefore, neglecting the second order terms when comparing with unity, one finds

$$n \cong \left(-\frac{\partial \zeta}{\partial r}\right)_T \ell$$

$$m = \left[\frac{\partial \zeta}{r \partial \theta} \frac{\partial \zeta}{\partial r} - \left(\frac{\partial \zeta}{\partial r}\right)^2 \frac{\partial \delta}{r \partial \theta} - \frac{\partial \delta}{r \partial \theta}\right]_T \ell$$

Third and last, the relation of directional cosines states that

$$m^2 + n^2 + \ell^2 = 1$$

Again neglecting the second order terms when comparing with unity, one finds:

$$l \cong \pm \left[\frac{1}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2} + 2 \frac{\partial h^*}{\partial R} \frac{\partial F}{\partial R}} \right]_C$$

$$m \cong \mp \left[\frac{\left\{ \frac{\partial F}{\partial R} + \frac{\partial G}{\partial Z} \left[1 + \left(\frac{\partial F}{\partial R}\right)^2 \right] \right\} \frac{\partial h^*}{R \partial \theta}}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2} + 2 \frac{\partial h^*}{\partial R} \frac{\partial F}{\partial R}} \right]_C$$

$$n \cong \mp \left[\frac{\frac{\partial F}{\partial R} + \frac{\partial h^*}{\partial R}}{\sqrt{1 + \left(\frac{\partial F}{\partial R}\right)^2} + 2 \frac{\partial h^*}{\partial R} \frac{\partial F}{\partial R}} \right]_C$$

since $\delta^* = \delta/a \cong \left(h^* \frac{\partial G}{\partial Z} \right)_C$ for small interface motion.